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EXTENSION OF THE LEVI-CIVITA THEOREM TO NONHOLONOMIC SYSTEMS*

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The Levi-Civita theorem on stationary solutions of an autonomous canonical system which admits invariant relations in involution is extended to nonholonomic systems with time-independent constraints. This was obtained using the canonical form of Voronets' equations. It is shown that the system can be extended to gyroscopic systems.

1. The canonical form of Voronets' equations. Consider a material system whose position is defined by n Lagrangian coordinates q_j and subjected to the action of forces that are derivatives of function $U(q_1, \ldots, q_n)$, and to the nonholonomic relations

$$q_r^{\cdot} = \sum_{i=1}^{k} b_{ri}(q_1, \ldots, q_n) q_i^{\cdot} \quad (k < n, \quad r = k+1, \ldots, n) \tag{1.1}$$

We denote the system kinetic energy by $T(q_i, q_r, q_i^{\cdot}, q_r^{\cdot})$ and set

$$\Theta(q_i, q_r, q_i^{\cdot}) = T\left(q_i, q_r, q_i^{\cdot}, \sum_{l=1}^{k} b_{rl} q_l^{\cdot}\right) (i = 1, \ldots, k, r = k+1, \ldots, n)$$

The equations of motion are obtained by supplementing Eqs.(1.1) by the Voronets equations /1/

$$\frac{d}{dt}\left(\frac{\partial\Theta}{\partial q_i^{\cdot}}\right) - \frac{\partial(\Theta+U)}{\partial q_i} - \sum_{r=k+1}^{n} \frac{\partial(\Theta+U)}{\partial q_r} b_{ri} = \sum_{r=k+1}^{n} \frac{\partial T}{\partial q_r^{\cdot}} \sum_{l=1}^{k} A_{il}^{(r)} q_l^{\cdot} \quad (i = 1, \ldots, k) \tag{1.2}$$

$$A_{il}^{(r)} = \frac{\partial b_{ri}}{\partial q_l} + \sum_{s=k+1}^{n} \frac{\partial b_{ri}}{\partial q_s} b_{sl} - \frac{\partial b_{rl}}{\partial q_i} - \sum_{s=k+1}^{n} \frac{\partial b_{rl}}{\partial q_s} b_{si}$$

where the derivatives $\partial T/\partial q_r^{\cdot}$ are expressed in terms of $q_i, q_r, q_i^{\cdot}$, and the quantities $A_{il}^{(r)}$ are antisymmetric with respect to indices i and l.

We set

$$L(q_i, q_r, q_i^{\cdot}) = \Theta + U, \; p_i = \partial L/\partial q_i^{\cdot}$$
$$H(q_i, q_r, p_i) = \sum_{i=1}^{k} p_i q_i^{\cdot} - L(q_i, q_r, q_i^{\cdot})$$

and write the Voronets equation in its canonical form

$$dq_i/dt = \partial H/\partial p_i \; (i = 1, \ldots, k) \tag{1.3}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} - \sum_{r=k+1}^{n} b_{ri}\frac{\partial H}{\partial q_r} + \sum_{r=k+1}^{n} \frac{\partial T}{\partial q_r^{\cdot}} \sum_{l=1}^{k} A_{il}^{(r)} \frac{\partial H}{\partial p_l}$$

where the derivatives $\partial T/\partial q_r^{\cdot}$ are expressed in terms of q_i, q_r, p_i. Equation (1.3) is to be supplemented by the equation of constraints

$$q_r^{\cdot} = \sum_{i=1}^{k} b_{ri}(q_1, \ldots, q_n) \frac{\partial H}{\partial p_i} \quad (r = k+1, \ldots, n) \tag{1.4}$$

In what follows the symbol D/Dt will denote the time derivative by virtue of system (1.3), (1.4).

Remarks.1°. Function H is the first integral of system (1.3), (1.4). Indeed, taking into consideration (1.4) and the antisymmetry of quantities $A_{il}^{(r)}$, we have $DH/Dt = 0$.

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2°. The necessary and sufficient condition for function $\varphi(q_i, q_r, p_i)$ to be the first integral of system (1.3), (1.4) is of the form

$$(\varphi, H)_k + \sum_{i=1}^{k}\sum_{r=k+1}^{n} b_{ri}\left(\frac{\partial\varphi}{\partial q_r}\frac{\partial H}{\partial p_i} - \frac{\partial\varphi}{\partial p_i}\frac{\partial H}{\partial q_r}\right) + \sum_{r=k+1}^{n}\sum_{i=1}^{k}\sum_{l=1}^{k}\frac{\partial T}{\partial q_r}A_{il}^{(r)}\frac{\partial\varphi}{\partial p_i}\frac{\partial H}{\partial p_l} = 0 \qquad (1.5)$$

$$(\varphi, H)_k = \sum_{i=1}^{k}\left(\frac{\partial\varphi}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial\varphi}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

3°. The condition of stationarity of H is

$$\frac{\partial H}{\partial q_i} = 0,\ \frac{\partial H}{\partial p_i} = 0,\ \frac{\partial H}{\partial q_r} = 0 \quad (i = 1, \ldots, k, r = k+1, \ldots, n)$$

These relations evidently constitute a set of invariant relations for system (1.3), (1.4), as shown by a direct test that

$$\frac{D}{Dt}\left(\frac{\partial H}{\partial q_i}\right),\quad \frac{D}{Dt}\left(\frac{\partial H}{\partial p_i}\right),\quad \frac{D}{Dt}\left(\frac{\partial H}{\partial q_r}\right)$$

are linear combinations of $\partial H/\partial q_i$, $\partial H/\partial p_i$, $\partial H/\partial q_r$ and, consequently, vanish together with them.

2. Extension of the Levi-Civita theorem to nonholonomic systems. Levi-Civita had shown /2,3/ that, when an autonomous canonical system has m invariant relations (respectively, m first integrals) which are in involution, it has ∞^m (respectively ∞^{2m}) particular solutions (called stationary) obtained by the integration of m first order differential equations of standard form. Let us extend that theorem to system (1.3), (1.4). Assume that system (1.3), (1.4) has m independent of time t invariant relations

$$f_u(q_1, \ldots, q_n, p_1, \ldots, p_k) = 0 \quad (u = 1, \ldots, m, m < k)$$

which satisfy conditions similar to (1.5)

$$(f_u, f_v)_k + \sum_{r=k+1}^{n}\sum_{i=1}^{k} b_{ri}\left(\frac{\partial f_u}{\partial q_r}\frac{\partial f_v}{\partial p_i} - \frac{\partial f_u}{\partial p_i}\frac{\partial f_v}{\partial q_r}\right) + \sum_{r=k+1}^{n}\sum_{i=1}^{k}\sum_{l=1}^{k}\frac{\partial T}{\partial q_r}A_{il}^{(r)}\frac{\partial f_u}{\partial p_i}\frac{\partial f_v}{\partial p_l} = 0 \quad (u, v = 1, \ldots, m) \qquad (2.1)$$

Suppose that the m invariant relations are solvable for $p_1, \ldots, p_m$

$$p_\alpha - \varphi_\alpha(q_1, \ldots, q_n, p_{m+1}, \ldots, p_k) = 0 \quad (\alpha = 1, \ldots, m) \qquad (2.2)$$

Taking into account the relations

$$\frac{\partial f_u}{\partial q_s} + \sum_{\alpha=1}^{m}\frac{\partial f_u}{\partial p_\alpha}\frac{\partial\varphi_\alpha}{\partial q_s} = 0 \quad (u = 1, \ldots, m, \quad s = 1, \ldots, n)$$

$$\frac{\partial f_v}{\partial p_h} + \sum_{\beta=1}^{m}\frac{\partial f_v}{\partial p_\beta}\frac{\partial\varphi_\beta}{\partial q_h} = 0 \quad (v = 1, \ldots, m, \quad h = m+1, \ldots, k)$$

we transform condition (2.1) to

$$\sum_{\alpha=1}^{m}\sum_{\beta=1}^{m}\frac{\partial f_u}{\partial p_\alpha}\frac{\partial f_v}{\partial p_\beta}F_{\alpha\beta}(q_1, \ldots, q_n, p_{m+1}, \ldots, p_k) = 0$$

Since the functional determinant of functions f_u is by assumption nonzero relative to p_α, these conditions reduce to

$$F_{\alpha\beta} = 0 \quad (\alpha, \beta = 1, \ldots, m)$$

Using the implicit expressions for $F_{\alpha\beta}$ we represent conditions (2.1) in the form

$$\frac{\partial\varphi_\alpha}{\partial q_\beta}-\frac{\partial\varphi_\beta}{\partial q_\alpha}+\{\varphi_\alpha,\varphi_\beta\}-\sum_{r=k+1}^{n}\left(b_{r\alpha}\frac{\partial\varphi_\beta}{\partial q_r}-b_{r\beta}\frac{\partial\varphi_\alpha}{\partial q_r}\right)-\sum_{r=k+1}^{n}\sum_{h=m+1}^{k}b_{rh}\left(\frac{\partial\varphi_\alpha}{\partial q_r}\frac{\partial\varphi_\beta}{\partial p_h}-\frac{\partial\varphi_\alpha}{\partial p_h}\frac{\partial\varphi_\beta}{\partial q_r}\right)- \tag{2.3}$$

$$\sum_{r=k+1}^{n}\frac{\partial T}{\partial \dot q_r}\left[A^{(r)}_{\alpha\beta}-\sum_{h=m+1}^{k}\left(A^{(r)}_{\alpha h}\frac{\partial\varphi_\beta}{\partial p_h}+A^{(r)}_{h\beta}\frac{\partial\varphi_\alpha}{\partial p_h}\right)+\sum_{h=m+1}^{k}\sum_{j=m+1}^{k}A^{(r)}_{hj}\frac{\partial\varphi_\alpha}{\partial p_h}\frac{\partial\varphi_\beta}{\partial p_j}\right]=0 \quad (\alpha,\beta=1,\ldots,m)$$

$$\{\varphi_\alpha,\varphi_\beta\}=\sum_{h=m+1}^{k}\left(\frac{\partial\varphi_\alpha}{\partial p_h}\frac{\partial\varphi_\beta}{\partial p_h}-\frac{\partial\varphi_\alpha}{\partial q_h}\frac{\partial\varphi_\beta}{\partial p_h}\right)$$

We differentiate the invariant relations $p_\alpha=\varphi_\alpha=0$ with respect to t on the basis of Eqs.(1.3) and (1.4), and obtain

$$\frac{\partial H}{\partial q_\alpha}+\{H,\varphi_\alpha\}+\sum_{\beta=1}^{m}\frac{\partial\varphi_\alpha}{\partial q_\beta}\frac{\partial H}{\partial p_\beta}+\sum_{r=k+1}^{n}\left(b_{r\alpha}-\sum_{h=m+1}^{k}b_{rh}\frac{\partial\varphi_\alpha}{\partial p_h}\right)\frac{\partial H}{\partial q_r}+\sum_{r=k+1}^{n}\frac{\partial\varphi_\alpha}{\partial q_r}\left[\sum_{\beta=1}^{m}b_{r\beta}\frac{\partial H}{\partial p_\beta}+\sum_{h=m+1}^{k}b_{rh}\frac{\partial H}{\partial p_h}\right]+ \tag{2.4}$$

$$\sum_{r=k+1}^{n}\sum_{l=1}^{k}\frac{\partial T}{\partial \dot q_r}\left(\sum_{h=m+1}^{k}A^{(r)}_{hl}\frac{\partial\varphi_\alpha}{\partial p_h}-A^{(r)}_{\alpha h}\right)\frac{\partial H}{\partial p_l}=0\ , \quad (\alpha=1,\ldots,m)$$

Using the notation

$$K(q_1,\ldots,q_n,p_{m+1},\ldots,p_k)=H(q_1,\ldots,q_n,\varphi_1,\ldots,\varphi_m,p_{m+1},\ldots,p_k)$$

we have

$$\frac{\partial K}{\partial q_\omega}=\frac{\partial H}{\partial q_\omega}+\sum_{\beta=1}^{m}\frac{\partial H}{\partial p_\beta}\frac{\partial\varphi_\beta}{\partial q_\omega} \tag{2.5}$$
$$\frac{\partial K}{\partial p_h}=\frac{\partial H}{\partial p_h}+\sum_{\beta=1}^{m}\frac{\partial H}{\partial p_\beta}\frac{\partial\varphi_\beta}{\partial p_h}$$
$$(\omega=1,\ldots,n,\ h=m+1,\ldots,k)$$

from which

$$\frac{\partial K}{\partial q_\alpha}+\{K,\varphi_\alpha\}=\frac{\partial H}{\partial q_\alpha}+\{H,\varphi_\alpha\}+\sum_{\beta=1}^{m}\frac{\partial H}{\partial p_\beta}\left[\frac{\partial\varphi_\beta}{\partial p_\alpha}-\{\varphi_\alpha,\varphi_\beta\}\right]$$
$$(\alpha=1,\ldots,m)$$

hence relations (2.4) with allowance for (2.3) and (2.5) assume the form

$$\frac{\partial K}{\partial q_\alpha}+\{K,\varphi_\alpha\}+\sum_{r=k+1}^{n}\left(b_{r\alpha}-\sum_{h=m+1}^{k}b_{rh}\frac{\partial\varphi_\alpha}{\partial p_h}\right)\frac{\partial K}{\partial q_r}+\sum_{r=k+1}^{n}\sum_{h=m+1}^{k}b_{rh}\frac{\partial\varphi_\alpha}{\partial q_r}\frac{\partial K}{\partial p_h}-\sum_{r=k+1}^{n}\sum_{h=m+1}^{k}\frac{\partial T}{\partial \dot q_r}\times \tag{2.6}$$

$$\left[A^{(r)}_{\alpha h}-\sum_{j=m+1}^{k}A^{(r)}_{jh}\frac{\partial\varphi_\alpha}{\partial p_j}\right]\frac{\partial K}{\partial p_h}=0 \quad (\alpha=1,\ldots,m)$$

where it is assumed that the coefficients are defined in terms of $q_1,\ldots,q_n,p_{m+1},\ldots,p_k$ by (1.4) and (2.2). Function K thus satisfies the system of Eqs.(2.6) in partial derivatives.

Let us show that the system consisting of Eqs.(2.2) and equations

$$\frac{\partial K}{\partial p_j}=0,\quad \frac{\partial K}{\partial q_j}=0,\quad \frac{\partial K}{\partial q_r}=0 \tag{2.7}$$
$$(j=m+1,\ldots,k,\quad r=k+1,\ldots,n)$$

represents a set of invariant relations for the system (1.3), (1.4).

First, using (2.5), we obtain

$$\frac{D}{Dt}\left(\frac{\partial K}{\partial p_j}\right)=\left\{K,\frac{\partial K}{\partial p_j}\right\}+\sum_{r=k+1}^{n}\sum_{h=m+1}^{k} b_{rh}\left(\frac{\partial^2 K}{\partial p_j \partial q_r}\frac{\partial K}{\partial p_h}-\frac{\partial^2 K}{\partial p_j \partial p_h}\frac{\partial K}{\partial q_r}\right)+\sum_{r=k+1}^{n}\sum_{h=m+1}^{k}\sum_{s=m+1}^{k}\frac{\partial T}{\partial q_r^{\cdot}}A_{hs}^{(r)}\frac{\partial^2 K}{\partial p_j \partial p_h}\frac{\partial K}{\partial p_s}+$$
$$\sum_{\alpha=1}^{m}\frac{\partial H}{\partial p_\alpha}\left[\frac{\partial^2 K}{\partial p_j \partial q_\alpha}-\left\{\varphi_\alpha\frac{\partial K}{\partial p_j}\right\}+\sum_{r=k+1}^{n}\frac{\partial^2 K}{\partial p_j \partial q_r}\times\left(b_{r\alpha}-\sum_{h=m+1}^{k} b_{rh}\frac{\partial \varphi_\alpha}{\partial p_h}\right)+\sum_{r=k+1}^{n}\sum_{h=m+1}^{k} b_{rh}\frac{\partial^2 K}{\partial p_j \partial p_h}\frac{\partial \varphi_\alpha}{\partial q_r}+\right.$$
$$\left.\sum_{r=k+1}^{n}\sum_{h=m+1}^{k}\frac{\partial T}{\partial q_r^{\cdot}}\left[A_{h\alpha}^{(r)}-\sum_{s=m+1}^{k}A_{hs}^{(r)}\frac{\partial \varphi_\alpha}{\partial p_s}\right]\frac{\partial^2 K}{\partial p_j \partial p_h}\right.$$

Differentiation of the system of Eqs.(2.6) with respect to p_j shows that the coefficient at $\partial H/\partial p_\alpha$ is a linear combination of first order derivatives of K. Finally, we have

$$\frac{D}{Dt}\left(\frac{\partial K}{\partial p_i}\right)=\left\{K,\frac{\partial K}{\partial p_j}\right\}+\sum_{r}\sum_{h} b_{rh}\left(\frac{\partial^2 K}{\partial p_j \partial q_r}\frac{\partial K}{\partial p_h}-\frac{\partial^2 K}{\partial p_j \partial p_h}\frac{\partial K}{\partial q_r}\right)+\sum_{r}\sum_{h}\sum_{s}\frac{\partial T}{\partial q_r^{\cdot}}A_{hs}^{(r)}\frac{\partial^2 K}{\partial p_j \partial p_h}\frac{\partial K}{\partial p_s}+$$
$$\sum_{\alpha}\frac{\partial H}{\partial p_\alpha}\left[\left\{\frac{\partial \varphi_\alpha}{\partial p_j},K\right\}-\sum_{r}\frac{\partial}{\partial p_j}\left[b_{r\alpha}-\sum_{h} b_{rh}\frac{\partial \varphi_\alpha}{\partial p_h}\right]\frac{\partial K}{\partial q_r}-\sum_{r}\sum_{h}\frac{\partial}{\partial p_j}\left(b_{rh}\frac{\partial \varphi_\alpha}{\partial q_r}\right)\frac{\partial K}{\partial p_h}+\right.$$
$$\left.\sum_{r}\sum_{h}\frac{\partial}{\partial p_j}\left[\frac{\partial T}{\partial q_r^{\cdot}}\left(A_{\alpha h}^{(r)}-\sum_{s}A_{hs}^{(r)}\frac{\partial \varphi_\alpha}{\partial p_s}\right)\right]\frac{\partial K}{\partial p_h}\right]$$

Hence (D/Dt) $(\partial K/\partial p_j)$ is a linear combination of $\partial K/\partial p_j, \partial K/\partial q_j, \partial K/\partial q_r$ $(j=m+1,\ldots, k;\ r=k+1,\ldots, n)$ and, consequently, vanishes together with them. The proof that the derivatives $(D/Dt)(\partial K/\partial q_j)$ and $(D/Dt)(\partial K/\partial q_r)$ are zero is similar, which proves the result. It follows from Eqs.(2.6) that Eqs.(2.7) yield $\partial K/\partial q_\alpha = 0$ $(\alpha = 1,\ldots, m)$, so that the $n+k-2m$ relations of (2.7) represent the condition of stationarity of K.

Thus, if system (1.3), (1.4) has m invariant relations that are solvable for $p_1,\ldots, p_m$ and satisfy conditions (2.1) or (2.3), the $n+k$ conditions of stationarity of H reduce to $n+k-2m$ equations (2.7).

Suppose that relations (2.7) are solvable for $q_{m+1},\ldots, q_k, q_{k+1},\ldots, q_n, p_{m+1},\ldots, p_k$. Then Eqs.(2.6) and (2.7) enable us to express these variables in terms of function of $q_1,\ldots, q_m$.

We divide system (1.3), (1.4) in two parts

$$\frac{dq_h}{dt}=\frac{\partial H}{\partial p_h}\quad (h=m+1,\ldots,k);\quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}-\sum_{r} b_{ri}\frac{\partial H}{\partial q_r}+\sum_{r}\sum_{l}\frac{\partial T}{\partial q_r^{\cdot}}A_{il}^{(r)}\frac{\partial H}{\partial p_l}$$
$$q_r^{\cdot}=\sum_{r} b_{ri}\frac{\partial H}{\partial p_i}\quad (i=1,\ldots,k;\ r=k+1,\ldots,n)$$
$$\frac{dq_\alpha}{dt}=\frac{\partial H}{\partial p_\alpha}\quad (\alpha=1,\ldots,m)$$

The first $2n-m$ equations necessarily satisfy the considered here values $q_{m+1},\ldots, q_n, p_1, \ldots, p_k$; by substituting in $\partial H/\partial p_\alpha$ for them functions of $q_1,\ldots, q_m$ we obtain a system of first order differential equations of standard form which are used for determining $q_1,\ldots, q_m$ as functions of t, and of m constants of integration. Substitution of the first integral for any of the invariant relations results in the appearance of a new constant.

We have thus obtained the following extension of the Levi-Civita theorem to nonholonomic systems.

Theorem. If system (1.3), (1.4) admits m invariant relations (respectively, m first integrals) solvable for m parameters $p_1,\ldots, p_m$ and satisfies conditions (2.1) or (2.3), then it has ∞^m (respectively, ∞^{2m}) particular solutions that are determined using m first order equations of standard form.

When there is only a single invariant relation $(m=1)$, conditions (2.1) or (2.3) are evidently automatically satisfied.

3. Example. Consider a heavy sphere of radius a and mass m, whose center of mass G coincides with its geometric center, and the mass is symmetrically distributed relative to the diameter Gz. Let C be the sphere moment of inertia about Gz and A the moment of inertia about the diameter normal to Gz. The sphere rolls without slipping on the horizontal plane $x_1O_1y_1$. We denote by O_1z the upward directed vertical with unit vector z_1. As parameters we have coordinates x, y of point G and Euler's angles θ, φ, ψ.

Let $\mathbf{V}_G$ be the velocity of point G, and Ω and σ be the sphere instantaneous angular velocity and moment of momentum about point G, respectively. The equations of motion are

$$\mathbf{V}_G = a\Omega \times \mathbf{z}_1 \tag{3.1}$$

$$\sigma + ma^2\mathbf{z}_1 \times (\Omega \times \mathbf{z}_1) = \mathbf{c} \tag{3.2}$$

where $\mathbf{c}$ is the vector constant of integration. The energy integral is of the form

$$\sigma \cdot \Omega + ma^2 (\Omega \times \mathbf{z}_1)^2 = h$$

Combining it with Eq.(3.2) we obtain

$$\Omega \cdot \mathbf{c} = h \tag{3.3}$$

Let $(c_0, 0, c_1)$ be components of $\mathbf{c}$ (by a suitable selection of axis O_1x_1 the second component can be reduced to zero). Formula (3.3) of form

$$f \equiv c_0 (\theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\theta\sin\psi) + c_1 (\psi^{\cdot} + \varphi^{\cdot}\cos\theta) - h = 0 \tag{3.4}$$

is used as the invariant relation.

Integration of the problem reduces to quadratures /4,5/, We shall show that the above theorem enables us to obtain particular solutions.

Equation (3.1) yields

$$x^{\cdot} - a(\theta^{\cdot}\sin\psi - \varphi^{\cdot}\sin\theta\cos\psi) = 0, \; y^{\cdot} + a(\theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\theta\sin\psi) = 0 \tag{3.5}$$

hence

$$2\Theta = (A + ma^2)\theta^{\cdot 2} + (A\psi^{\cdot 2} + ma^2\varphi^{\cdot 2})\sin^2\theta + C(\psi^{\cdot}\cos\theta + \varphi^{\cdot})^2$$

(in this case Voronets' equations reduce to Chaplygin's equations /1/).

We denote by $p_\theta, p_\varphi, p_\psi$ the variables conjugates of θ, φ, ψ and obtain

$$H = \frac{p_\theta^2}{2(A + ma^2)} + \frac{1}{2\Delta}\{(A\sin^2\theta + C\cos^2\theta)\, p_\varphi^2 - 2\cos\theta p_\varphi p_\psi + (C + ma^2\sin^2\theta)\, p_\psi^2$$

$$\Delta = [CA + ma^2(A\sin^2\theta + C\cos^2\theta)]\sin^2\theta$$

and (3.4) assumes the form

$$f \equiv \frac{1}{\Delta}\{(c_0\sin\theta\sin\psi + c_1\cos\theta)\,[(A\sin^2\theta + C\cos^2\theta)\, p_\varphi - C\cos\theta p_\psi] + c_1[-C\cos\theta p_\varphi + (C + ma^2\sin^2\theta)\, p_\psi]\} + \frac{c_0 p_\theta\cos\psi}{A + ma^2} - h = 0 \tag{3.6}$$

With condition (3.6) satisfied, the conditions of stationarity of H are written in symbolic form as $\delta H - \lambda\delta f = 0$, where λ is an undetermined multiplier. From this

$$-\theta^{\cdot}\sin\psi + \varphi^{\cdot}\sin\theta\cos\psi = 0 \tag{3.7}$$

$$p_\theta = \lambda c_0\cos\psi, \quad p_\varphi = \lambda(c_0\sin\theta\sin\psi + c_1\cos\theta), \quad p_\psi = \lambda c_1 \tag{3.8}$$

Representing (3.2) in the form of its projection on the axis O_1y_1 and taking into account (3.7) we obtain

$$\psi^{\cdot}\cos\theta + \varphi^{\cdot} = 0 \tag{3.9}$$

which implies that

$$p_\varphi = ma^2\sin^2\theta\varphi^{\cdot}, \quad p_\psi = A\sin^2\theta\psi^{\cdot}$$

Hence using (3.8) we have

$$\frac{p_\varphi}{p_\psi} = \frac{ma^2}{A}\frac{d\varphi}{d\psi} = \frac{c_0}{c_1}\sin\theta\sin\psi + \cos\theta$$

and, finally, taking into account (3.9) we obtain

$$\operatorname{tg}\theta = \frac{\gamma}{\sin\psi} \quad \left(\gamma = -\frac{A + ma^2}{A}\frac{c_1}{c_0}\right) \tag{3.10}$$

Formulas (3.9) and (3.10) yield

$$\frac{d\varphi}{d\psi} = \pm\frac{\sin\psi}{\sqrt{\gamma^2 + \sin^2\psi}}$$

hence (Φ is the constant of integration)

$$\cos(\varphi - \Phi) = \frac{\cos\psi}{\sqrt{\gamma^2+1}} \tag{3.11}$$

Differentiation of (3.10) with respect to time yields

$$\theta^{\cdot} = -\frac{\gamma\cos\psi}{\gamma^2+\sin^2\psi}\psi^{\cdot}$$

Using this formula and also (3.9) and (3.4) we obtain for ψ the differential equation

$$\frac{\psi^{\cdot}}{\gamma^2+\sin^2\psi} = h' \quad \left(h' = h\left[c_0\gamma\left(1+\frac{\gamma^2 A}{A+ma^2}\right)\right]^{-1}\right) \tag{3.12}$$

Consequently

$$\operatorname{ctg}\psi = -\sqrt{\frac{\gamma^2+1}{\gamma}}\operatorname{tg}\left[h'\gamma\sqrt{\gamma^2+1}\,(t-\tau)\right] \tag{3.13}$$

where τ is the new constant of integration. Conditions (3.5) with allowance for (3.7), (3.9), and (3.12) assume the form

$$x^{\cdot} = 0, \quad y^{\cdot} = a\gamma h'$$

i.e. point G moves uniformly along a straight line parallel to O_1y_1.

Thus ∞^5 motions of the sphere have been indicated; similar results were obtained by Agostinelli /5/.

4. One generalization. Let us now show that the Levi-Civita theorem can be extended to a material system whose position is determined by n Lagrangian coordinates $q_1,\ldots,q_n$, and which is subjected to forces defined by derivatives of function $U(q_1,\ldots,q_n)$ and, also, to gyroscopic forces (this result was obtained in /6/ for systems of a less general form and, because of an error in calculations, only for particular cases).

Motions of this system are defined by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{\cdot}}\right) - \frac{\partial L}{\partial q_i} = \sum_{k=1}^{n} g_{ik}q_k^{\cdot} \quad (i=1,\ldots,n)$$

where $L(q_i, q_i^{\cdot})$ is the Lagrangian, and g_{ik} are continuously differentiable functions of $q_1,\ldots, q_k, q_1^{\cdot},\ldots, q_k^{\cdot}$ which satisfy conditions $g_{ik} = -g_{ki}$ $(i, k = 1,\ldots, n)$.

Setting

$$p_i = \frac{\partial L}{\partial q_i^{\cdot}}, \quad H = \sum_{i=1}^{n} p_i q_i^{\cdot} - L^{\cdot}$$

shows that the equations of motion can be represented in the canonical form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + \sum_{k=1}^{n} g_{ik}\frac{\partial H}{\partial p_k} \quad (i=1,\ldots,n) \tag{4.1}$$

where g_{ik} is now expressed in terms of q_i, p_i.

Assume that system (4.1) has m independent of time t invariant relations

$$f_u(q_1,\ldots, q_n, p_1,\ldots, p_n) = 0 \quad (u = 1,\ldots, m < n)$$

that satisfy the conditions

$$(f_u, f_v) + \sum_{r=1}^{n}\sum_{s=1}^{n} g_{rs}\frac{\partial f_u}{\partial p_r}\frac{\partial f_v}{\partial p_s} = 0 \quad (u, v = 1,\ldots, m) \tag{4.2}$$

where (f_u, f_v) are Poisson's brackets of f_u and f_v.

Let us carry out calculations as in Sect.2, and assume that m invariant relations are solvable for $p_1,\ldots, p_m$

$$p_\alpha - \varphi_\alpha(q_1,\ldots, q_n, p_{m+1},\ldots, p_n) = 0 \quad (\alpha = 1,\ldots, m) \tag{4.3}$$

Conditions (4.2) assume the form

$$\frac{\partial\varphi_\alpha}{\partial q_\beta}-\frac{\partial\varphi_\beta}{\partial q_\alpha}+\{\varphi_\alpha,\varphi_\beta\}-g_{\alpha\beta}+\sum_{h=m+1}^{n}\left(g_{\alpha h}\frac{\partial\varphi_\beta}{\partial p_h}+g_{h\beta}\frac{\partial\varphi_\alpha}{\partial p_h}\right)-\sum_{h=m+1}^{n}\sum_{s=m+1}^{n}g_{hs}\frac{\partial\varphi_\alpha}{\partial p_h}\frac{\partial\varphi_\beta}{\partial p_s}=0\quad(\alpha=1,\ldots,m)\qquad(4.4)$$

$$\{\varphi_\alpha,\varphi_\beta\}=\sum_{j=m+1}^{n}\left(\frac{\partial\varphi_\alpha}{\partial p_j}\frac{\partial\varphi_\beta}{\partial q_j}-\frac{\partial\varphi_\alpha}{\partial q_j}\frac{\partial\varphi_\beta}{\partial p_j}\right)$$

Differentiating with respect to time relations (4.3) with allowance for (4.1) and introducing the function

$$K(q_1,\ldots,q_n,p_{m+1},\ldots,p_n)=H(q_1,\ldots,q_n,\varphi_1,\ldots,\varphi_m,p_{m+1},\ldots,p_n)$$

we obtain

$$\frac{\partial K}{\partial q_\alpha}+\{K,\varphi_\alpha\}-\sum_{j=m+1}^{n}\left[g_{\alpha h}-\sum_{j=m+1}^{n}g_{jh}\frac{\partial\varphi_\alpha}{\partial p_j}\right]\frac{\partial K}{\partial p_h}=0\qquad(\alpha=1,\ldots,m)\qquad(4.5)$$

where $g_{\alpha h}$ and g_{jh} are expressed in terms of $q_1,\ldots,q_n,p_{m+1},\ldots,p_n$ using (4.3). Hence K satisfies the system of differential equations with partial derivatives (4.5).

Moreover, it is possible to show that Eqs.(4.3) and

$$\partial K/\partial p_j=0,\ \partial K/\partial q_j=0\quad(j=m+1,\ldots,n)\qquad(4.6)$$

yield a set of invariant relations for system (4.1). For example, we have

$$\frac{D}{Dt}\left(\frac{\partial K}{\partial p_j}\right)=\left\{K,\frac{\partial K}{\partial p_j}\right\}+\sum_{h=m+1}^{n}\sum_{s=m+1}^{n}g_{hs}\frac{\partial^2K}{\partial p_j\partial p_h}\frac{\partial K}{\partial p_s}+$$
$$\sum_{\alpha=1}^{m}\frac{\partial H}{\partial p_\alpha}\left[\left\{\frac{\partial\varphi_\alpha}{\partial p_j},K\right\}+\sum_{h=m+1}^{n}\frac{\partial}{\partial p_j}\left(g_{\alpha h}-\sum_{s=m+1}^{n}g_{sh}\frac{\partial\varphi_\alpha}{\partial p_s}\right)\frac{\partial K}{\partial p_h}\right]$$

hence $(D/Dt)(\partial K/\partial p_j)$ is a linear combination of first partial derivatives of K.

It follows from Eqs.(4.5) that Eqs.(4.6) imply that $\partial K/\partial q_\alpha=0$ $(\alpha=1,\ldots,m)$. Because of this the conditions of stationarity of K reduce to $2(n-m)$ conditions (4.6). If we assume that Eqs.(4.6) are solvable for q_h,p_h $(h=m+1,\ldots,n)$, then Eqs.(4.3) and (4.6) enable us to express $q_{m+1},\ldots,q_n,p_1,\ldots,p_n$ in terms of functions of $q_1,\ldots,q_m$. Substituting these into the equations

$$dq_\alpha/dt=\partial H/\partial p_\alpha\quad(\alpha=1,\ldots,m)$$

we obtain $q_1,\ldots,q_m$ in the form of functions of t, and m constants of integration.

Theorem. If system (4.1) has m invariant relations (respectively, m first integrals) solvable for m parameters p_i and satisfy condition (4.2) or (4.4), it has ∞^m (respectively, ∞^{2m}) particular solutions that are obtained as the result of integration of m first order equations of standard form.

The last theorem includes, as a particular case, the extension of the Levi-Civita theorem to nonholonomic systems defined by Chaplygin's equations.

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